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鉛直チャンネル中の自然対流におけるスパン方向運動量生成について

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概要

We investigate the bifurcation of three-dimensional tertiary flows numerically in a simple shear layer with a cubic velocity profile when secondary flow loses its stability to oscillatory perturbations. It is found that the bifurcating motion is either of periodic nature or of travelling-wave nature, depending on the spanwise symmetry of disturbances. Furthermore, it turns out that the travelling-wave propagating in the spanwise direction generates the spanwise mean flow.

1 Introduction

It is of considerable importance to applications in engineering and geophysics, among others, to understand the mechanism of the transition from laminar flow to early stages of turbulence in plane parallel shear layers. As a simple example of such shear layers we consider flows with a cubic velocity profile. These flows with an inflectional velocity profile can be realized between two parallel vertical plates which are kept at constant different temperatures under the gravity field. The flows are characterised by a upward motion near a hotter plate and by a downward motion near a colder plate, so that the momentum for the undisturbed state is only in the vertical direction. It is well known that Squire's theorem is applicable in this case, so that it is sufficient to analyse the stability of the basic state with respect to two-dimensional (spanwise-independent) perturbations. In fact, a spanwise-independent secondary flow characterized by cats' eye-like transverse vortices sets in as the shear gets stronger (Vest & Arpaci[1]). The stability analysis on the secondary flow indicates that the secondary flow becomes unstable to three-dimensional perturbations with either a monotone subharmonic nature or an oscillatory harmonic nature (Nagata & Busse[2]). In the present paper we investigate the nonlinear development of the perturbations in the oscillatory harmonic case using two numerical schemes: a direct numerical simulation to integrate the time evolution of the primitive variables for Navier-Stokes equations and a Newton-Raphson iterative scheme to solve the nonlinear algebraic equations for the disturbance amplitudes in an equilibrium state.

2 Mathematical formulation

We consider a viscous incompressible fluid motion between two parallel vertical plates of infinite extent separated by the distance $2d$. We take the origin of a Cartesian coordinate system on the mid-plane between the plates with the x^*- and y^*-axes along the plate and the z^*-axis in the direction normal to the plate. The unit vectores, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, are associated in those directions.

The basic flow with a cubic velocity profile can be realised by taking a Boussinesq fluid between the plates maintained at constant different temperatures, T_+ and T_-. The conductive state is represented by a linear temperature variation across the fluid layer, and buoyancy balances the viscous force (see Fig.1).

By taking an appropriate nondimensionalisation the basic state (U_B:the velocity and T_B: the temperature) is described by

$$U_B(z) = G_r\, z(1-z^2) \quad \text{and} \quad T_B(z) = z \tag{1}$$

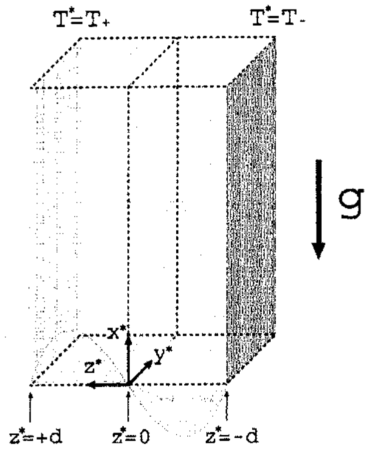

図 1: Configuration

G_r in (1) is the Grashof number defined by $G_r = \gamma g d^3(T_+ - T_-)/(2\nu^2)$ where γ is the coefficient of thermal expansion, ν is the kinematic viscosity and g is the acceleration due to gravity.

The equations which govern disturbances deviated from the basic state (1) are written by

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0\,, & (2)\\ \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= -\nabla p + \theta\hat{\boldsymbol{i}} + \nabla^2\boldsymbol{u}\,, & (3)\\ \partial_t \theta + (\boldsymbol{u}\cdot\nabla)\theta &= \frac{1}{Pr}\nabla^2\theta\,, & (4) \end{aligned}$$

where **u** is the velocity disturbance, p the pressure disturbance, θ the temperature disturbance, $Pr = \nu/\kappa$ the Prandtl number. The no-slip boundary condition and the fixed temperatures are prescribed on the plates:

$$\boldsymbol{u} = 0 \ \text{ and } \ \theta = 0 \ \text{ at } \ z = \pm 1. \tag{5}$$

It may be easily found that temperature disturbnce becomes identically zero in the vanishing Prandtl number limit.

The stability of the basic state is governed by

$$\partial_t \nabla^2 \Delta_2 \tilde{\phi} + \{U_B(z)\partial_x - \nabla^2\}\nabla^2\Delta_2\tilde{\phi} - U_B''(z)\partial_x\Delta_2\tilde{\phi} = 0, \tag{6}$$

and

$$\partial_t \Delta_2 \tilde{\psi} + \{U_B(z)\partial_x - \nabla^2\}\Delta_2\tilde{\psi} - U_B'(z)\partial_y\Delta_2\tilde{\phi} = 0, \tag{7}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are the poloidal and toroidal components of an infinitesimal velocity perturbation $\tilde{\boldsymbol{u}}$:

$$\tilde{\boldsymbol{u}} = \nabla \times \nabla \times (\tilde{\phi}\boldsymbol{k}) + \nabla \times (\tilde{\psi}\boldsymbol{k}). \tag{8}$$

superimposed on the basic flows (1).

We express $\tilde{\phi}$ and $\tilde{\psi}$ as

$$\tilde{\phi} = \sum_{\ell=0}^{\infty} \tilde{a}_l \left(1 - z^2\right)^2 T_l(z) \exp\{i\alpha x + i\beta y + \sigma t\}, \tag{9}$$

$$\tilde{\psi} = \sum_{ell=0}^{\infty} \tilde{b}_l \left(1 - z^2\right) T_l(z) \exp\{i\alpha x + i\beta y + \sigma t\}, \tag{10}$$

where α and β are the wavenumbers in the streamwise and the spanwise directions, respectively. $T_\ell(z)$ is the ℓ-th Chebyshev polynomial.

In order to analyse the nonlinear development of the perturbation we consider a velocity deviation $\hat{\boldsymbol{u}}$ from the laminar state and for convenience separate it into the average part $\check{U}(z)\boldsymbol{i} + \check{V}(z)\boldsymbol{j}$ and the residual $\check{\boldsymbol{u}}$, so that the total velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = U(z)\boldsymbol{i} + \check{V}(z)\boldsymbol{j} + \check{\boldsymbol{u}}, \tag{11}$$

where $U(z) = U_B(z) + \check{U}(z)$. The residual $\check{\boldsymbol{u}}$ is further decomposed into the poloidal and toroidal parts:

$$\check{\boldsymbol{u}} = \nabla \times \nabla \times (\phi \boldsymbol{k}) + \nabla \times (\psi \boldsymbol{k}). \tag{12}$$

The nonlinear state is governed by

$$\begin{aligned} \partial_t \nabla^2 \triangle_2 \phi &+ \{U(z)\partial_x - \nabla^2\}\nabla^2\triangle_2\phi - U''(z)\partial_x\triangle_2\phi \\ &+ \check{V}(z)\partial_y\nabla^2\triangle_2\phi - \check{V}''(z)\partial_y\triangle_2\phi \\ &+ \delta[(\check{\boldsymbol{u}}\cdot\nabla)\check{\boldsymbol{u}}] = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} \partial_t \triangle_2 \psi &+ \{U(z)\partial_x - \nabla^2\}\triangle_2\psi - U'(z)\partial_y\triangle_2\phi \\ &+ \check{V}(z)\partial_y\triangle_2\psi + \check{V}'(z)\partial_x\triangle_2\phi \\ &+ \epsilon[(\check{\boldsymbol{u}}\cdot\nabla)\check{\boldsymbol{u}}] = 0, \end{aligned} \tag{14}$$

$$\check{U}'' + \partial_z \overline{\triangle_2\phi(\partial^2_{xz}\phi + \partial_y\psi)} = \partial_t\check{U}, \tag{15}$$

$$\check{V}'' + \partial_z \overline{\triangle_2\phi(\partial^2_{yz}\phi - \partial_x\psi)} = \partial_t\check{V}, \tag{16}$$

where the differential operators ϵ in (11) and δ in (12) are defined by

$$\epsilon \equiv \boldsymbol{k}\cdot(\nabla\times \quad \text{and} \quad \delta \equiv \boldsymbol{k}\cdot(\nabla\times\nabla\times\,. \tag{17}$$

and the overline in (15) or (16) stands for the x, y-average. In the above, the possibility of induced average flow $\check{V}(z)$ in the spanwise direction is incorporated. We express ϕ, ψ, $\check{U}$ and $\check{V}$ as follows:

$$\begin{aligned} \phi &= \sum_{l=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} a_{lmn} \left(1 - z^2\right)^2 T_l(z) \\ &\quad \times \exp(im\alpha(x - c_x t) + in\beta(y - c_y t)) \end{aligned} \tag{18}$$

$$\begin{aligned}
\psi &= \sum_{l=0}^{\infty} \sum_{\substack{m=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} b_{lmn} \left(1-z^2\right) T_l(z) \\
&\quad \times \exp(im\alpha(x-c_x t) + in\beta(y-c_y t)) &(19)\\
\check{U} &= \sum_{l=0}^{\infty} c_l \left(1-z^2\right) T_l(z) &(20)\\
\check{V} &= \sum_{l=0}^{\infty} d_l \left(1-z^2\right) T_l(z). &(21)
\end{aligned}$$

In the expressions (18)-(19) the phase velocities, c_x and c_y, are included in order to deal with a travelling-wave type of nonlinear equilibrium states. For a steady state time-derivatives are omitted in the equations (13)- (16) and c_x and c_y are both zero. As a measure of nonlinearity we choose the momentum transport τ on the plates normalised by its value for the basic state:

$$\tau = U'(z)/U'_B(z)|_{z=\pm 1} \tag{22}$$

In order to investigate the stability of the equilibrium state, we superimpose arbitrary three-dimensional infinitesimal perturbations on the nonlinear state (18) - (21). The stability equations linearised with respect to the perturbations are given by

$$\begin{aligned}
\partial_t \nabla^2 \triangle_2 \tilde{\phi} &+ \{U(z)\partial_x - \nabla^2\}\nabla^2 \triangle_2 \tilde{\phi} - U''(z)\partial_x \triangle_2 \tilde{\phi} \\
&+ \check{V}(z)\partial_y \nabla^2 \triangle_2 \tilde{\phi} - \check{V}''(z)\partial_y \triangle_2 \tilde{\phi} \\
&+ \delta[(\check{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{u}}\cdot\nabla)\check{\boldsymbol{u}}] = 0, &(23)
\end{aligned}$$

$$\begin{aligned}
\partial_t \triangle_2 \tilde{\psi} &+ \{U(z)\partial_x - \nabla^2\}\triangle_2 \tilde{\psi} - U'(z)\partial_y \triangle_2 \tilde{\phi} \\
&+ \check{V}(z)\partial_y \triangle_2 \tilde{\psi} + \check{V}'(z)\partial_x \triangle_2 \tilde{\phi} \\
&+ \epsilon[(\check{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{u}}\cdot\nabla)\check{\boldsymbol{u}}] = 0. &(24)
\end{aligned}$$

$\tilde{\phi}$ and $\tilde{\psi}$ are expressed by

$$\begin{aligned}
\tilde{\phi} &= \sum_{l=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{a}_{lmn} \left(1-z^2\right)^2 T_l(z) \\
&\times \exp\{i(m\alpha+d)(x-c_x t) + i(n\beta+b)(y-c_y t) \quad + \quad \sigma t\}, &(25)\\
\tilde{\psi} &= \sum_{l=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{b}_{lmn} \left(1-z^2\right) T_l(z) \\
&\times \exp\{i(m\alpha+d)(x-c_x t) + i(n\beta+b)(y-c_y t) \quad + \quad \sigma t\}, &(26)
\end{aligned}$$

where d and b are Floquet parameters.

3 Numerical methods

3.1 Stability and bifurcation analysis

Substitution of the expansions (18) - (21) into the basic equations (13) - (16) reduces to a set of nonlinear algebraic equations for the expansion coefficients with the aid of the Chebyshev collocation method. We employ the Newton-Raphson method to solve the set of equations.

For the stability of an nonlinear equilibrium state we superimpose the general form of a three-dimensional perturbation (25) and (26) on the equilibrium state and we obtain the eigenvalue problem for the growth rate $\sigma$of the perturbation as the eigenvalue with the aid of Floquet's theorem.

3.2 Direct numerical simulation

We choose the wall-normal components of the velocity and the vorticity in addition to the mean velocity components in the streamwise and the spanwise directions as the flow field. The flow field is expanded by the Fourier series in the streamwise and spanwise directions and the Chebyshev polynomials in the wall-normal direction. The time-development of the flow is followed by the method of the Adams-Bashforth scheme for convective terms and the Crank-Nicolson scheme for viscous terms.

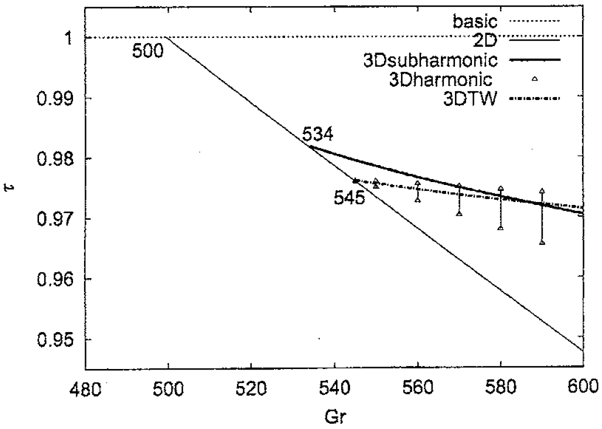

図 2: The bifurcation diagram

4 Results

The bifurcation diagram in the (G_r, τ) space is shown in Fig. 2. The basic flow becomes unstable at $G_r = 500$ to a two-dimensional perturbation and 2D transverse vortex flow (2DTV) with the streamwise wavenumber $\alpha = 1.25$ as a secondary flow bifurcates supercritically. The 2DTV becomes unstable first at $G_r = 534$ to a steady 3D subharmonic three-dimensional perturbation with the Floquet parameters $(d, b) = (0.625 = \alpha/2, 1.0)$, and later at $G_r = 545$ to a harmonic three-dimensional perturbation with $(d, b) = (0, 1.0)$. The real eogenvalues are associated with the subharmonic perturbation and the the complex conjugate pair of eigenvalues with the harmonic perturbation. We obtain a 3D steady subharmonic flow (3DSS) which bifurcates at $G_r = 534$ as a tertiary solution as shown by the thick cueve in Fig. 2. Time-dependent solutions as a tertiary state may occur at $G_r = 545$. For DNS we restricted the wavenumber pair (α, β) for the computation domain $(L_x = 2\pi/\alpha L_y = 2\pi/\beta)$ to (1.25, 1.00) for the harmonic case and to (0.625, 1.0) for the subharmonic case. Since the three-dimensional subharmonic solution cannot manifest itself by DNS in the harmonic domain we expect some three-dimensional periodic flows to bifurcate directly from the 2DTV at $G_r = 545$ in the harmonic case. However, it turns out that the periodic flows exist only as a transient state and the solution in the final state is actually a three-dimensional travelling-wave (3DTW) instead. The 3DTW does not change its flow pattern in a frame moving with the spanwise phase speed c_y and keeps a constant momentum transport on the plates as indicated by the dashed curve in Fig. 2. The existence of the 3DTW is also confirmed by the calculation by Newton-Raphson method. It is interesting to note that the 3DTW has a non-zero average velocity $\check{V}(z)$ in the spanwise direction. It should be noted that the solutions in the harmonic case constitute a subset of the solutions in the subharmonic case.

5 Conclusions

In the present paper we have investigated the nonlinear development of the perturbations in the oscillatory harmonic case using two numerical schemes, a direct numerical simulation and a Newton-Raphson iterative scheme. Both numerical schemes have indicated that the bifurcating three-dimensional flow when the secondary flow loses its stability to an oscillatory disturbance is periodic for the case where the spanwise symmetry is retained, whereas it is of travelling-wave type travelling with a constant phase in the spanwise direction when the spanwise symmetry is broken. The mean flow produced by nonlinear interactions of oscillatory perturbations has only the streamwise component for the three-dimensional periodic flow. It turns out that the mean flow has an additional spanwise component, thus generating the spanwise momentum, for the three-dimensional travelling-wave solution. We will also show the generation of the spanwise momentum by means of symmetry arguments.

In experiments the vertical fluid layer between two plates must be confined by the side walls, in general, which ought to inhibit the total mass flux in the horizontal direction. Therefore, in order to detect a horizontal mass flux we plan to carry out an experiment on na natural convection in a vertical annulus. The mass flux sould be generated in the azimuthal direction, either clockwise or anti-clockwise depending on the form of an initial disturbance.

参考文献

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